UDC 534

ON NONLINEAR PERTURBATIONS UNDER RESONANCE

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N. V. VOROB'EV
(Moscow)
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We study the properties of the solutions of one two-frequency system in the whole space. We establish the correspondence between its solutions and the phase trajectories of an autonomous second-order system obtained from the original one by using its group properties. The original system can be treated as a model for the study of the phenomena taking place in multifrequency systems under resonance. This paper is closely related to the investigations in [1].

1. We consider the following autonomous system of differential equations:

$$dz_1 / d\tau = z_4, \quad dz_3 / d\tau = (z_1^2 + z_2^2)(\beta z_2 + \alpha z_1)$$
$$dz_2 / d\tau = z_3, \quad dz_4 / d\tau = (z_1^2 + z_2^2)(\beta z_1 - \alpha z_2) \qquad (1.1)$$
$$(\alpha^2 + \beta^2 \neq 0)$$

System (1.1) is evolutionary and describes the behavior of the original systems at times $t \sim O(\sqrt{\varepsilon^{-1}})$. Here system (1.1) will be transformed to a form which permits us to investigate it in the whole phase space. By a direct check we convince ourselves that (1.1) is invariant relative to a group of transformations, namely, simultaneous rotation through angle ϑ in the ($z_1 z_2$)- and ($z_4 z_3$)-planes. Consequently, the number of unknown functions in (1.1) can be diminished by one [2], for which we need to find the invariants of the infinitesimal operator of the group and to adopt them as the new variables. For the group indicated the operator and its invariants have the form

$$M \equiv z_2 \frac{\partial}{\partial z_1} - z_1 \frac{\partial}{\partial z_2} - z_4 \frac{\partial}{\partial z_3} + z_3 \frac{\partial}{\partial z_4}$$

$$c_1 = z_1^2 + z_2^2, \qquad c_2 = z_3^2 + z_4^2, \qquad c_3 = \frac{z_1 z_3 - z_2 z_4}{z_1 z_4 + z_2 z_3}$$

We introduce the new unknown functions

$$r = \sqrt{z_1^2 + z_2^2}, \qquad \rho = \sqrt{z_3^2 + z_4^2}, \qquad \operatorname{tg} \varphi = \frac{z_1 z_3 - z_2 z_4}{z_1 z_4 + z_2 z_3} \qquad (1.2)$$

For the mapping of space Z into the space of r, ρ, φ to be single-valued we set $r \geqslant 0$, $\rho \geqslant 0$, $\varphi \in [0, 2\pi]$. Then, if $r(\tau)$ or $\rho(\tau)$ vanish at some instant, $\varphi(\tau)$ changes by π by a jump at this point. Transformation (1.2) is continuous at the remaining points of the phase space. In the new variables system (1.1) is invariant relative to the following group of dilations:

$$r' = \lambda r, \quad \rho' = \lambda^2 \rho, \quad \tau' = \lambda^{-1}\tau \quad (\lambda > 0)$$

We introduce new variables $R = \rho / r^2$, $dt = r d\tau$ and we pass to the Cartesian coordinates $x = R \sin \varphi$, $y = R \cos \varphi$. Then system (1.1) takes the form

$$dx / dt = \alpha - 3xy, \qquad dy / dt = \beta + x^2 - 2y^2 \qquad (1.3)$$

Thus, as a corollary of system (1.1) we have obtained the second-order system (1.3) whose phase trajectories are investigated below by standard methods.

2. If in system (1.3) we make a scale transformation of the variables $x = x_* x'$, $y = y_* y'$, $t = t_* t'$, then by an appropriate choice of the constants x_*, y_*, t_* we can decrease the number of parameters in the right hand side of (1.3) to one. Therefore, below, in those cases when this is convenient, instead of the α and β occurring in (1.3) we shall use the notation $\sin\psi$ and $\cos\psi$. We note that system (1.3) remains invariant under the substitution $x \to (-x)$, $y \to (-y)$, $t \to (-t)$. Consequently, the phase trajectory pattern is symmetrical about the origin, but the direction of motion along symmetric trajectories is different, therefore, if a singular point is stable, the one symmetric to it is unstable.

Analysis shows that two singular points are located in the finite part of the plane for all values of α and β $(\alpha^2 + \beta^2 \neq 0)$, and for $\alpha > 0$ the singular points lie in the first and third quadrants, while for $\alpha < 0$, in the second and fourth quadrants. A singular point located in the upper part of the phase plane is always stable, while in the lower part this point is unstable. Depending on the values of parameter $\psi \in [0, 2\pi]$ the singular points are of the following types: for $\psi \in [0, \psi_0)$ and $\psi \in (2\pi - \psi_0, 2\pi]$ they are nodes, for $\psi = \psi_0$ and $\psi = 2\pi - \psi_0$ they are logarithmic nodes, for $\psi \in (\psi_0, \pi)$ and $\psi \in (\pi, 2\pi - \psi_0)$ they are foci, and for $\psi = \pi$ they are centers. Here $\psi_0 = \mathrm{arctg}\,(6\sqrt{6}/47) \approx 18^\circ$.

On the (xy)-plane, for any value of ψ the singular points are located on the closed curve

$$4y_0^4 + 5x_0^2y_0^2 + x_0^4 = 1$$

surrounding the origin, moreover, for $|y_0| > 2\sqrt{6}\,|x_0|$ the singular point is a node, for $|y_0| = 2\sqrt{6}\,|x_0|$ it is a logarithmic node, for $|y_0| < 2\sqrt{6}\,|x_0|$ it is a focus, and for $y_0 = 0$ it is a center. The points at infinity are not stationary singular points for system (1.3), but for any value of parameter ψ there exists a unique trajectory passing through the point at infinity located at the "end" of the y-axis. When $\sin\psi \neq 0$ this phase trajectory is asymptotically representable, as $y \to \pm\infty$, by the formula

$$x = (\sin\psi)/5y + O(1/y^3) \tag{2.1}$$

For $\sin\psi = 0$ this trajectory coincides identically with the y-axis.

The study of the singular points allows us to obtain the phase trajectory pattern only locally in the neighborhood of these points. It is convenient to begin to ascertain the behavior of the phase trajectories on the whole plane with the case $\psi = \pi$ which corresponds to $\alpha = 0$, $\beta < 0$. In this case the general integral of system (1.3) has the form

$$x^2 + y^2 - \beta/2 = cx^{4/5} \tag{2.2}$$

From (2.2) it follows that in the case $\psi = \pi$ the phase plane xy of system (1.3) is entirely filled by closed nested curves surrounding the singular points. The y-axis is a closed trajectory passing through the point at infinity. In Fig. 1 a we show the qualitative behavior of the phase trajectories in this case (recall that the trajectories are symmetric about the origin, but that the directions of motion on symmetric trajectories are opposite).

In the general case of $\psi \neq \pi$ system (1.3) is not integrable, and to construct its full phase pattern it is necessary to clear up the question of the existence of limit cycles in it [3]. From the nature of the singular points of system (1.3) it follows that any of the limit cycles, if they exist, can surround only one of the singular points. There will not automatically be limit cycles around a singular point if we succeed in showing that the trajectory passing through the point at infinity tends to this singular point as $t \to +\infty$

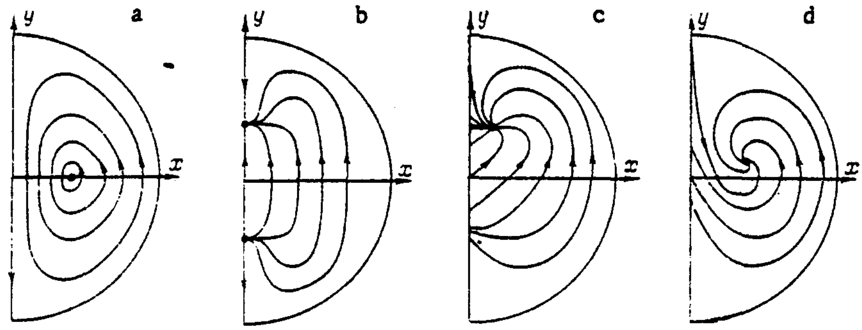

Fig. 1

or as $t \to -\infty$. To prove this we apply a method based on the use of Liapunov function which we will construct using the general integral (2.2) found for $\psi = \pi$. Let an equation $H(x, y, c) = 0$ yield a family of simple closed nonintersecting curves surrounding a singular point and filling up some region G. Let $x = x(t)$, $y = y(t)$ be a trajectory located in G. We consider the function $H(x(t), y(t), c)$. If its derivative computed relative to system (1.3) turns out to be negative definite in G, then any trajectory falling into this region tends to the singular point as $t \to +\infty$. The proof will be complete if we show that the trajectory passing through the point at infinity falls into G.

Let us set $\alpha > 0$, then one of the system's singular points lies in the first quadrant. If in (1.3) we translate the origin to the singular point, we obtain

$$
\begin{aligned}
dx''/dt &= -3y_0x'' - 3y'' - 3x''y \\
dy''/dt &= 2x'' - 4y_0y'' + x''^2 - 2y''^2
\end{aligned}
\tag{2.3}
$$

Here $y_0 \geqslant 0$ is the ordinate of the system's singular point. For $y_0 = 0$ system (2.3) has the general integral

$$H'' \equiv x''^2 + 2x'' + y''^2 + 3/2 = c\,(x'' + 1)^{4/3} \qquad (c \geqslant 3/2)$$

This family of closed nonintersecting curves surrounds the origin and fills the halfplane $x'' > -1$. Let us find the total derivative of H'' relative to Eqs. (2.3),

$$W = \frac{dH''}{dt} = \frac{-2y_0\,(x'' + 2)\,(x''^2 + 2y''^2)}{(x'' + 1)^{7/3}}$$

Obviously, $W < 0$ for $x'' > -1$, $y \in (-\infty, \infty)$. By the same token we have shown that any trajectory of system (1.3) falling into the right halfplane tends to the singular point. From formula (2.1) it follows that the trajectory passing through the point at infinity necessarily falls into the right halfplane. The proof of the absence of limit cycles in system (1.3) when $\psi \neq \pi$ is completed.

We can now describe the pattern of the behavior of the phase trajectories of system (1.3). As $t \to +\infty$ all trajectories approach the singular point located in the upper halfplane, while as $t \to -\infty$, they approach the point in the lower halfplane. Figure 1b shows the behavior of the trajectories for $\psi = 0$. Figure 1c shows the general form of trajectory behavior in the case of a node. The behavior of the phase trajectories when the singular points are foci is shown in Fig. 1d. The number of zeros of function $y(t)$ corresponding to an arbitrarily taken phase trajectory differs by no more than four from the number of points of intersection with the x-axis of the phase trajectory passing through the origin. From linear theory we can obtain an estimate for the number of intersections with the x-axis of the integral trajectory passing through the origin,

$$N \sim \frac{2\sqrt{6}}{7\pi} |\operatorname{ctg} \psi| \quad \text{as} \quad \psi \to \pi$$

Let us describe the behavior of functions $x(t)$, $y(t)$ when the singular points of system (1.3) are not centers. In this case all trajectories going from one singular point to the other and remaining in the finite part of the phase plane are defined for $t \in (-\infty, \infty)$ and have the coordinates of the singular points as their limit values. The functions $x(t)$, $y(t)$ corresponding to the phase trajectory passing through the point at infinity (it is not a singular point for system (1.3)) have, in a neighborhood of this point, the following asymptotic representation:

$$y \sim {}^1/_2\,(t - t_0)^{-1}, \qquad x \sim {}^2/_5\,\alpha\,(t - t_0) \tag{2.4}$$

Consequently, they also are defined for $t \in (-\infty, \infty)$, and $y(t)$ has a first-order pole at $t = t_0$.

3. From system (1.3) we establish the connection of the functions $z_1(\tau)$ and $z_2(\tau)$ of system (1.1) with the functions $x(t)$ and $y(t)$ whose behavior is now known. Consider the identity

$$(z_1^2 + z_2^2)\, z_4 = (z_1 z_4 + z_2 z_3)\, z_1 - (z_1 z_3 - z_2 z_4)\, z_2$$

It is valid for arbitrary z_1, z_2, z_3, z_4. If, however, these functions are solutions of system (1.1), then with the aid of the transformations considered in Sect. 1 we can obtain the following equation for $z_1(t)$:

$$dz_1 / dt = y z_1 \pm x\sqrt{r^2 - z_1^2}$$

The general solution of this equation has the form

$$z_1 = r(t) \sin \left(\int_0^t x(\vartheta)\, d\vartheta + c_1 \right) \tag{3.1}$$

Analogously, from the identity

$$(z_1^2 + z_2^2)\, z_3 = (z_1 z_4 + z_2 z_3)\, z_2 + (z_1 z_3 - z_2 z_4)\, z_1$$

we obtain

$$z_2 = r(t) \cos \left(\int_0^t x(\vartheta)\, d\vartheta + c_1 \right) \tag{3.2}$$

From formulas (3.1), (3.2) it follows that the behavior of the solutions of system (1.1) is completely determined by the functions $x(t)$ and $y(t)$ given by (1.3) if we know the connection between t and τ, defined by the equation $dt = r d\tau$.

Knowing the behavior of $y(t)$ from (1.3), we can ascertain the nature of $r(t)$,

$$r(t) = r_0 \exp \left(\int_0^t y(\vartheta)\, d\vartheta \right) \tag{3.3}$$

Hence it follows that for bounded $y(t)$ the function $r(t)$ is strictly positive, while if $y(t)$ has a singularity of form (2.4) at $t = t_0$, then $r(t) \sim \sqrt{|t - t_0|}$ in the neighborhood of this point. From (3.3) it follows that for $y(t) \not\equiv$ const the function $r(t)$ grows exponentially as $t \to \pm\infty$. It is obvious that $r(t)$ has local extrema at the points where the phase trajectory intersects the x-axis.

The function $\tau(t)$ is determined by the formula

$$\tau = \int_0^t \frac{d\vartheta}{r(\vartheta)} \tag{3.4}$$

Hence it follows that $\tau(t)$ defined for $t \in (-\infty, \infty)$, increases strictly monotonically, and, if $y(t) \not\equiv$ const, the function $\tau(t)$ is bounded, where its range of variation is of order $L \sim 1 / y_0$ $(y_0 > 0)$. In the neighborhood of the point $t = t_0$, where $r(t)$ vanishes, we have $\tau \sim \pm\sqrt{|t - t_0|}$. Consequently, the inverse function $t(\tau)$ defined on a finite interval, is monotonic and unbounded. Hence if follows that if $y(t) \not\equiv$ const, the composite function $r[t(\tau)]$ is nonnegative, is defined on a finite interval, becomes $+\infty$ (these are first order-poles) on the boundaries of its domain, and has as many local extrema as the times the phase trajectory $y(t)$ intersects the x-axis. If $r(\tau_0) = 0$, in the neighborhood of the point $\tau = \tau_0$ we have $r(\tau) \sim |\tau - \tau_0|$.

If $y(t) \equiv y_0 =$ const, the function $r(\tau)$ can be written out in explicit form,

$$r(\tau) = r_0 / (1 - y_0 r_0 \tau)$$

As we see from formulas (3.2) and (3.1) the function $r(\tau)$ describes the change in the amplitudes of the oscillations of functions $z_1(\tau)$ and $z_2(\tau)$. The form and the frequency of the oscillations depend upon the function

$$\Phi = \int_0^t x(\vartheta)\, d\vartheta.$$

As $t \to \infty$ we have $\Phi(t) \sim x_0 t$, hence for the independent variable τ the oscillation period tends to zero as τ tends to the boundary of the domain.

4. Let us consider the critical case when the singular points of system (1.3) are centers. In this case $x(t)$ and $y(t)$ are periodic functions. By virtue of the symmetry of the phase trajectories the function $y(t)$ has a mean value of zero over the period, consequently, the $r(t)$ determined by formula (3.3) is a periodic function. If $y(t)$ is bounded, $r(t)$ is a strictly positive function. In this case the function $\tau(t)$ from formula (3.4) is strictly monotonic and can be represented as

$$\tau(t) = gt + \varphi(t), \qquad g = \int_0^{T_0} \frac{d\vartheta}{z(\vartheta)}$$

where $\varphi(t)$ is some periodic function with the same period T_0 as $r(t)$. In this case the $r(\tau)$ obtained in parametric form from formulas (3.3), (3.4) is a periodic function with period $T_1 = gT_0$.

Indeed, let us fix an arbitrary τ_0. Because $\tau(t)$ is unbounded and strictly monotonic, we can find a unique $t_0 = t_0(\tau_0)$; for this value t_0 we find $r(t_0)$. Consider the values of the parameter $t_1 = t_0 + T_0$. Then $r(t_1) = r(t_0)$, $\tau_1 = g(t_0 + T_0) + \varphi(t_0) = \tau_0 + T_1$. Hence we obtain the equality $r(\tau + T_1) = r(\tau)$ for any τ.

It is obvious that period T_1 depends on the choice of the closed trajectory of system (1.3) on the (xy)-plane. The function $r(\tau)$ acquires a simpler form when the closed trajectory coincides with the y-axis. In this case $r(\tau)$ satisfies the equation $d^2r / d\tau^2 + 2r^3 = 0$; its general solution can be expressed in terms of elliptic functions. Although the function $r(\tau)$ has been obtained as periodic, the functions $z_1(\tau)$ and $z_2(\tau)$ are, in general, only almost periodic. This follows from the fact that a product of periodic functions with incommensurate periods occurs in the right-hand sides of (3.2) and (3.1).

5. Example. In system (1.1) let $\alpha = 0.2$, $\beta = 0.8$, then

$$\psi = \arcsin(\alpha / \sqrt{\alpha^2 + \beta^2}) = 76°$$

Consequently, the singular points of system (1.3) are foci and are located in the first and third quadrants. The domain of function $r(\tau)$ is of order $L \sim 1 / y_0 = 12$, on the boundaries of the domain $r(\tau)$ has first-order poles. The local extrema can be found by the approximate formula

$$N \simeq \frac{1}{7\pi} \sqrt{24\left(\frac{x_0}{y_0}\right)^2 - 1}\ \ln\left[\left(\frac{x_0}{y_0}\right)^2 - 1\right]$$

Here x_0, y_0 are the coordinates of the singular point. In the case being considered $x_0 = 0.9$, $y_0 = 0.08$, hence, $N = 3$.

Thus, we have ascertained the nature of the behavior of the solution of system (1.1) as a function of the values of the parameters occurring in the right-hand side. We have established that the system has a bounded general solution defined for $\tau \in (-\infty, \infty)$ only in exceptional cases when equality-type conditions are imposed on the parameters. In all remaining cases the system has an unbounded general solution defined on a finite interval, whose nature for any initial conditions is completely determined by the phase trajectory pattern of system (1.3).

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